SECTION 13.1: VECTORS IN THE PLANE

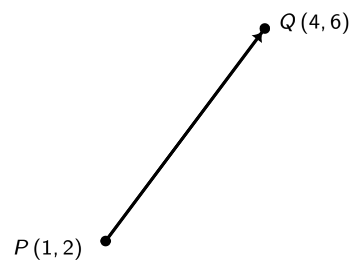
DEFINITION: A **vector** is a mathematical quantity which possesses both a **magnitude**, along with a **direction**.

Common quantities modeled by vectors include: velocities and forces.

A vector is represented geometrically as a 'directed' line segment.

- The 'magnitude' of the vector is taken to be the **length** of the line segment
- The direction of the vector is indicated by an **arrow** at one endpoint of the line segment.

Notationally, the symbol $\vec{v}$ is read as 'the vector v'.
Below is a typical vector $\vec{v}$ with endpoints $P(1,2)$ and $Q(4,6)$.
The point P is called the **initial point** or **tail** of $\vec{v}$ and the point Q is called the or **head** of $\vec{v}$.
Since we can reconstruct $\vec{v}$ completely from P and Q, we write $\vec{v} = \overrightarrow{PQ}$.
NOTE: The order of points P (initial point) and Q (terminal point) is important.

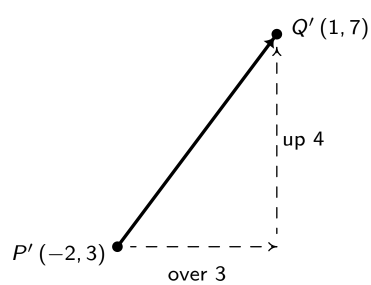


$\vec{v} = \overrightarrow{PQ}$

While it is true that P and Q completely determine $\vec{v}$, it is important to note that since vectors are defined in terms of their two characteristics, magnitude and direction, any directed line segment with the same length and direction as $\vec{v}$ is considered to be the same vector as $\vec{v}$, regardless of its initial point.

In the case of our vector $\vec{v}$ above, any vector which moves three units to the right and four up from its initial point to arrive at its terminal point is considered the same vector as $\vec{v}$. The notation we use to capture this idea is the **component form** of the vector, $\vec{v} = \langle 3, 4\rangle$, where the first number, 3, is called the x-**component** of $\vec{v}$ and the second number, 4, is called the y-**component** of $\vec{v}$.

For example, if we wanted to reconstruct $\vec{v} = \langle 3, 4\rangle$ with initial point $P'(-2,3)$, then we would find the terminal point of $\vec{v}$ by adding 3 to the x-coordinate and adding 4 to the y-coordinate to obtain the terminal point $Q'(1,7)$.



$\vec{v} = \langle 3, 4\rangle$ with initial point $P'(-2,3)$.

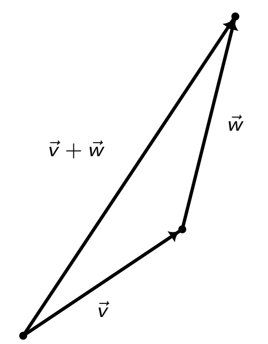
DEFINITION: Suppose $\vec{v}$ is represented by a directed line segment with initial point $P(x_0, y_0)$ and terminal point $Q(x_1, y_1)$. The **component form** of $\vec{v}$ is given by

$$\vec{v} = \overrightarrow{PQ} = \langle x_1 - x_0, y_1 - y_0 \rangle = \langle \Delta x, \Delta y \rangle$$

NOTE: Two vectors are equal if and only if their corresponding components are equal.

That is, $\langle v_1, v_2 \rangle = \langle v_1', v_2' \rangle$ if and only if $v_1 = v_1'$ and $v_2 = v_2'$.

ADDING VECTORS: Suppose we are given two vectors $\vec{v}$ and $\vec{w}$. The sum, or **resultant** vector $\vec{v} + \vec{w}$ is obtained as follows. First, plot $\vec{v}$. Next, plot $\vec{w}$ so that its initial point is the terminal point of $\vec{v}$. To plot the vector $\vec{v} + \vec{w}$ we begin at the initial point of $\vec{v}$ and end at the terminal point of $\vec{w}$. It is helpful to think of the vector $\vec{v} + \vec{w}$ as the 'net result' of moving along $\vec{v}$ then moving along $\vec{w}$.



$\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$

Our next step is to define addition of vectors component-wise to match the geometric action.

DEFINITION: Suppose $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$. The vector $\vec{v} + \vec{w}$ is defined by

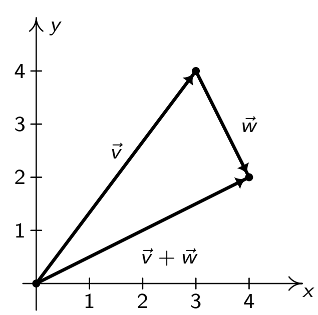
$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$$

EXAMPLE 1: Let $\vec{v} = \langle 3, 4 \rangle$ and let $\vec{w} = \overrightarrow{PQ}$ where $P(-3, 7)$ and $Q(-2, 5)$.
Find $\vec{v} + \vec{w}$ and interpret this sum geometrically.

SOLUTION: We first write $\vec{w}$ in component form: $\vec{w} = \langle -2 - (-3), 5 - 7 \rangle = \langle 1, -2 \rangle$. Thus,

$$\vec{v} + \vec{w} = \langle 3, 4 \rangle + \langle 1, -2 \rangle = \langle 3 + 1, 4 + (-2) \rangle = \langle 4, 2 \rangle .$$

We draw $\vec{v}$ with its initial point at $(0, 0)$ (for convenience) so that its terminal point is $(3, 4)$. Next, we graph $\vec{w}$ with its initial point at $(3, 4)$. Moving one to the right and two down, we find the terminal point of $\vec{w}$ to be $(4, 2)$. The vector $\vec{v} + \vec{w}$ has initial point $(0, 0)$ and terminal point $(4, 2)$ so its component form is $\langle 4, 2 \rangle$.



In order for vector addition to enjoy the same kinds of properties as real number addition, it is necessary to extend our definition of vectors to include a 'zero vector', $\vec{0} = \langle 0, 0 \rangle$.

Geometrically, $\vec{0}$ represents a point, which we can (very broadly) think of as a directed line segment with the same initial and terminal points. The reader may well object to the inclusion of $\vec{0}$, since after all, vectors are supposed to have both a magnitude (length) and a direction.

QUESTION: What is the magnitude of $\vec{0}$? What is the direction of $\vec{0}$?

PROPERTIES OF VECTOR ADDITION:

- **COMMUTATIVE PROPERTY:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.
- **ASSOCIATIVE PROPERTY:** For all vectors $\vec{u}, \vec{v}$ and $\vec{w}$, $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.
- **ADDITIVE IDENTITY:** For all vectors $\vec{v}$,

$$\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}.$$

  That is, the vector $\vec{0}$ acts as the additive identity for vector addition.

- **ADDITIVE INVERSE:** For every vector $\vec{v} = \langle v_1, v_2 \rangle$, the vector $\vec{w} = \langle -v_1, -v_2 \rangle$ satisfies

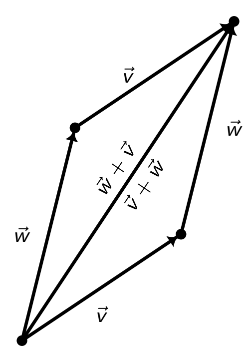
$$\vec{v} + \vec{w} = \vec{w} + \vec{v} = \vec{0}.$$

  That is, the additive inverse of a vector is the vector of the additive inverses of its components.

For the commutative property, we note that if $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ then

$$\begin{aligned} \vec{v} + \vec{w} &= \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle \\ &= \langle v_1 + w_1, v_2 + w_2 \rangle \\ &= \langle w_1 + v_1, w_2 + v_2 \rangle \\ &= \vec{w} + \vec{v} \end{aligned}$$

Geometrically, we can 'see' the commutative property by realizing that the sums $\vec{v} + \vec{w}$ and $\vec{w} + \vec{v}$ are the same directed diagonal determined by the parallelogram below.



Demonstrating the commutative property of vector addition.

The additive identity property is likewise verified algebraically using a calculation. If $\vec{v} = \langle v_1, v_2 \rangle$, then

$$\vec{v} + \vec{0} = \langle v_1, v_2 \rangle + \langle 0, 0 \rangle = \langle v_1 + 0, v_2 + 0 \rangle = \langle v_1, v_2 \rangle = \vec{v}.$$

From the commutative property of vector addition, we get that $\vec{0} + \vec{v} = \vec{v}$ as well.

QUESTION: What is the 'picture proof' of the additive identity property?

Regarding additive inverses, we can verify by direct computation that if $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle -v_1, -v_2 \rangle$,

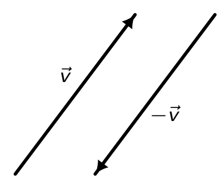
$$\vec{v} + \vec{w} = \langle v_1, v_2 \rangle + \langle -v_1, -v_2 \rangle = \langle v_1 + (-v_1), v_2 + (-v_2) \rangle =< 0, 0 >= \vec{0}.$$

Once again, the commutative property of vector addition assures us that, likewise, $\vec{w} + \vec{v} = \vec{0}$.

Moreover, additive inverses of vectors are **unique**. That is, given a vector $\vec{v} = \langle v_1, v_2 \rangle$, there is precisely only **one** vector $\vec{w}$ so that $\vec{v} + \vec{w} = \vec{0}$. To see this, suppose a vector $\vec{w} = \langle w_1, w_2 \rangle$ satisfies $\vec{v} + \vec{w} = \vec{0}$. By the definition of vector addition, we have $\langle v_1 + w_1, v_2 + w_2 \rangle = \langle 0, 0 \rangle$. Hence, $v_1 + w_1 = 0$ and $v_2 + w_2 = 0$. We get $w_1 = -v_1$ and $w_2 = -v_2$ so that $\vec{w} = \langle -v_1, -v_2 \rangle$.

Hence, every vector $\vec{v}$ has one, and only one, additive inverse. In general, we denote the additive inverse of a vector $\vec{v}$ with the (highly suggestive) notation $-\vec{v}$.

Geometrically, the vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $-\vec{v} = \langle -v_1, -v_2 \rangle$ have the same length, but opposite directions. As a result, when adding the vectors geometrically, the sum $\vec{v} + (-\vec{v})$ results in starting at the initial point of $\vec{v}$ and ending back at the initial point of $\vec{v}$. That is, the net result of moving $\vec{v}$ then $-\vec{v}$ is not moving at all.



Using the additive inverse of a vector, we can define the difference of two vectors: $\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$. Looking at this at the level of components, we see if $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ then

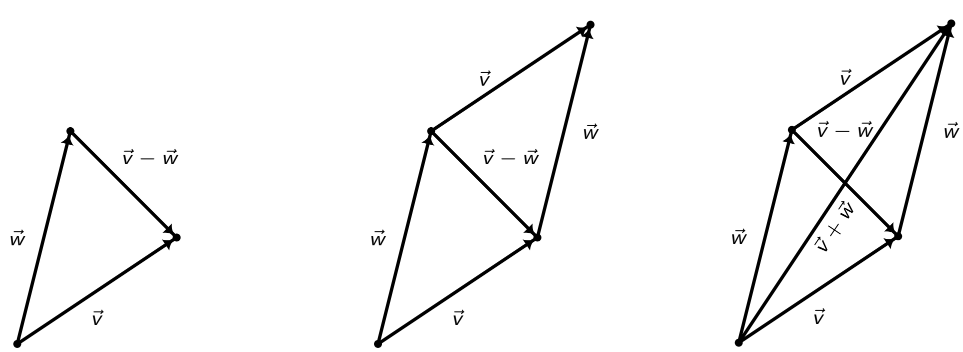
$$\begin{aligned} \vec{v} - \vec{w} &= \vec{v} + (-\vec{w}) \\ &= \langle v_1, v_2 \rangle + \langle -w_1, -w_2 \rangle \\ &= \langle v_1 + (-w_1), v_2 + (-w_2) \rangle \\ &= \langle v_1 - w_1, v_2 - w_2 \rangle \end{aligned}$$

In other words, like vector addition, vector subtraction works component-wise.

To interpret the vector $\vec{v} - \vec{w}$ geometrically, we note

$$\begin{aligned} \vec{w} + (\vec{v} - \vec{w}) &= \vec{w} + (\vec{v} + (-\vec{w})) && \text{Definition of Vector Subtraction} \\ &= \vec{w} + ((-\vec{w}) + \vec{v}) && \text{Commutativity of Vector Addition} \\ &= (\vec{w} + (-\vec{w})) + \vec{v} && \text{Associativity of Vector Addition} \\ &= \vec{0} + \vec{v} && \text{Definition of Additive Inverse} \\ &= \vec{v} && \text{Definition of Additive Identity} \end{aligned}$$

This means that the 'net result' of moving along $\vec{w}$ then moving along $\vec{v} - \vec{w}$ is just $\vec{v}$ itself. Geometrically, in the parallelogram determined by $\vec{v}$ and $\vec{w}$, the $\vec{v} - \vec{w}$ is one of the diagonals and the other is $\vec{v} + \vec{w}$.



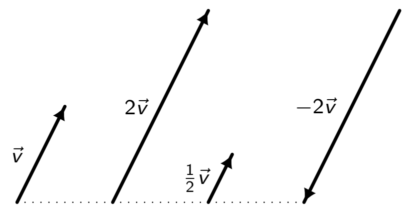
SCALAR MULTIPLICATION:

A **scalar** is just another word for 'real number.' In this context, we are multiplying a real number times a vector. We define this operation as the real number acting as a **scaling** factor of the vector.

DEFINITION: If k is a real number and $\vec{v} = \langle v_1, v_2 \rangle$, we define $k\vec{v}$ by

$$k\vec{v} = k\langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle$$

Scalar multiplication by k in vectors can be understood geometrically as scaling the vector (if $k > 0$) or scaling the vector and reversing its direction (if $k < 0$) as demonstrated below.



Note that $(-1)\vec{v} = (-1)\langle v_1, v_2 \rangle = \langle (-1)v_1, (-1)v_2 \rangle = \langle -v_1, -v_2 \rangle = -\vec{v}$, which is what we would expect.

PROPERTIES OF SCALAR MULTIPLICATION:

- **ASSOCIATIVE PROPERTY:** For every vector $\vec{v}$ and scalars k and r, $(kr)\vec{v} = k(r\vec{v})$.
- **MULTIPLICATIVE IDENTITY:** For all vectors $\vec{v}$, $1\vec{v} = \vec{v}$.
- **ADDITIVE INVERSE:** For all vectors $\vec{v}$, $-\vec{v} = (-1)\vec{v}$.
- **DISTRIBUTIVE PROPERTY OF SCALAR MULTIPLICATION OVER SCALAR ADDITION:** For every vector $\vec{v}$ and scalars k and r,

$$(k + r)\vec{v} = k\vec{v} + r\vec{v}$$

- **DISTRIBUTIVE PROPERTY OF SCALAR MULTIPLICATION OVER VECTOR ADDITION:** For all vectors $\vec{v}$ and $\vec{w}$ and scalars k,

$$k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$$

- **ZERO PRODUCT PROPERTY:** If $\vec{v}$ is vector and k is a scalar, then

$$k\vec{v} = \vec{0} \quad \text{if and only if} \quad k = 0 \quad \text{or} \quad \vec{v} = \vec{0}$$

These properties are true due to the definition of scalar multiplication and properties of real numbers.
For example, to prove the associative property, we let $\vec{v} = \langle v_1, v_2 \rangle$. If k and r are scalars then

$$\begin{aligned}
(kr)\vec{v} &= (kr)\langle v_1, v_2 \rangle \\
&= \langle (kr)v_1, (kr)v_2 \rangle && \text{Definition of Scalar Multiplication} \\
&= \langle k(rv_1), k(rv_2) \rangle && \text{Associative Property of Real Number Multiplication} \\
&= k\langle rv_1, rv_2 \rangle && \text{Definition of Scalar Multiplication} \\
&= k(r\langle v_1, v_2 \rangle) && \text{Definition of Scalar Multiplication} \\
&= k(r\vec{v})
\end{aligned}$$

You are encouraged to think this property means geometrically. The remaining properties are proved similarly.

DEFINITION: Two nonzero vectors $\vec{v}$ and $\vec{w}$ are called **parallel** if there is a (nonzero) scalar k so that $\vec{v} = k\,\vec{w}$. That is, **parallel** vectors have the **same** or **opposite** direction.

EXAMPLE 2: Which of the following vectors are parallel? $\vec{u} = < 1, -2 >$, $\vec{v} = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$, $\vec{w} = < -2, -4 >$?

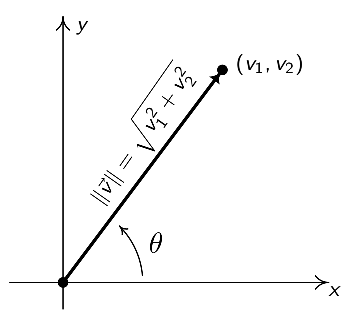
The properties of vector addition and scalar multiplication set up an algebra of vectors.
See if you can follow the reasoning of each step in the following example.

EXAMPLE 3: Solve $5\vec{v} - 2(\vec{v} + \langle 1, -2\rangle) = \vec{0}$ for $\vec{v}$.

SOLUTION:

$$\begin{aligned}
5\vec{v} - 2(\vec{v} + \langle 1,-2\rangle) &= \vec{0} \\
5\vec{v} + (-1)\left[2(\vec{v} + \langle 1,-2\rangle)\right] &= \vec{0} \\
5\vec{v} + [(-1)(2)](\vec{v} + \langle 1,-2\rangle) &= \vec{0} \\
5\vec{v} + (-2)(\vec{v} + \langle 1,-2\rangle) &= \vec{0} \\
5\vec{v} + [(-2)\vec{v} + (-2)\langle 1,-2\rangle] &= \vec{0} \\
5\vec{v} + [(-2)\vec{v} + \langle (-2)(1), (-2)(-2)\rangle] &= \vec{0} \\
[5\vec{v} + (-2)\vec{v}] + \langle -2, 4\rangle &= \vec{0} \\
(5 + (-2))\vec{v} + \langle -2, 4\rangle &= \vec{0} \\
3\vec{v} + \langle -2, 4\rangle &= \vec{0} \\
(3\vec{v} + \langle -2, 4\rangle) + (-\langle -2, 4\rangle) &= \vec{0} + (-\langle -2, 4\rangle) \\
3\vec{v} + [\langle -2, 4\rangle + (-\langle -2, 4\rangle)] &= \vec{0} + (-1)\langle -2, 4\rangle \\
3\vec{v} + \vec{0} &= \vec{0} + \langle (-1)(-2), (-1)(4)\rangle \\
3\vec{v} &= \langle 2, -4\rangle \\
\tfrac{1}{3}(3\vec{v}) &= \tfrac{1}{3}(\langle 2, -4\rangle) \\
\left[\left(\tfrac{1}{3}\right)(3)\right]\vec{v} &= \left\langle \left(\tfrac{1}{3}\right)(2), \left(\tfrac{1}{3}\right)(-4)\right\rangle \\
1\vec{v} &= \left\langle \tfrac{2}{3}, -\tfrac{4}{3}\right\rangle \\
\vec{v} &= \left\langle \tfrac{2}{3}, -\tfrac{4}{3}\right\rangle
\end{aligned}$$

DEFINITION: A vector whose initial point is $(0, 0)$ is said to be in **standard position**. If $\vec{v} = \langle v_1, v_2\rangle$ is plotted in standard position, then its terminal point is necessarily (v_1, v_2).



$\vec{v} = \langle v_1, v_2\rangle$ in standard position.

Recall the magnitude of vector $\vec{v}$ is the length of the directed line segment representing $\vec{v}$. When plotted in standard position, the length of this line segment is none other than the distance from the origin $(0, 0)$ to the point (v_1, v_2). Hence, the magnitude of $\vec{v}$, which we denote $\|\vec{v}\|$, is given by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$.

Turning to the notion of direction, we note that the point (v_1, v_2) is on the terminal side of the angle θ depicted in the diagram above. We have $v_1 = \|\vec{v}\|\cos(\theta)$ and $v_2 = \|\vec{v}\|\sin(\theta)$. From the definition of scalar multiplication and vector equality, we get

$$\begin{aligned}
\vec{v} &= \langle v_1, v_2 \rangle \\
&= \langle \|\vec{v}\|\cos(\theta), \|\vec{v}\|\sin(\theta) \rangle \\
&= \|\vec{v}\| \langle \cos(\theta), \sin(\theta) \rangle
\end{aligned}$$

This motivates the following definition.

DEFINITION: Suppose $\vec{v}$ is a vector with component form $\vec{v} = \langle v_1, v_2 \rangle$. Let θ be an angle in standard position whose terminal side contains the point (v_1, v_2).

- The **magnitude** of $\vec{v}$, denoted $\|\vec{v}\|$, is given by $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$
- If $\vec{v} \neq \vec{0}$, the **(vector) direction** of $\vec{v}$, denoted $\hat{v}$ is given by $\hat{v} = \langle \cos(\theta), \sin(\theta) \rangle$

Taken together, we get $\vec{v} = \|\vec{v}\| \langle \cos(\theta), \sin(\theta) \rangle = \|\vec{v}\|\hat{v} = \text{(magnitude)(direction)}$.

PROPERTIES OF MAGNITUDE AND DIRECTION: Suppose $\vec{v}$ is a vector.

- $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$
- For all scalars k, $\|k\,\vec{v}\| = |k|\|\vec{v}\|$.
- If $\vec{v} \neq \vec{0}$ then $\vec{v} = \|\vec{v}\|\hat{v}$, so that $\hat{v} = \left(\dfrac{1}{\|\vec{v}\|}\right)\vec{v}$.

Once again, the proofs here boil down to calculations. For example, If $\vec{v} = \langle v_1, v_2 \rangle$ and k is a scalar then

$$\begin{aligned}
\|k\,\vec{v}\| &= \|k\,\langle v_1, v_2 \rangle\| \\
&= \|\langle kv_1, kv_2 \rangle\| && \text{Definition of scalar multiplication} \\
&= \sqrt{(kv_1)^2 + (kv_2)^2} && \text{Definition of magnitude} \\
&= \sqrt{k^2 v_1^2 + k^2 v_2^2} \\
&= \sqrt{k^2(v_1^2 + v_2^2)} \\
&= \sqrt{k^2}\sqrt{v_1^2 + v_2^2} && \text{Product Rule for Radicals} \\
&= |k|\sqrt{v_1^2 + v_2^2} && \text{Since } \sqrt{k^2} = |k| \\
&= |k|\|\vec{v}\|
\end{aligned}$$

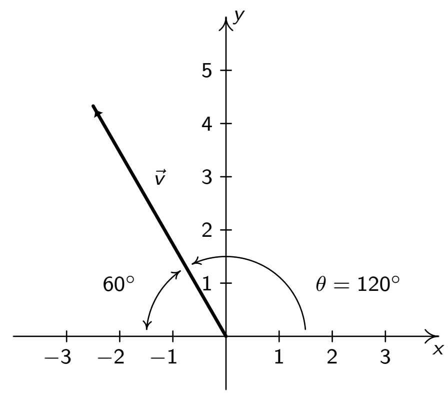
EXAMPLE 4:

1. Find the component form of the vector $\vec{v}$ with $\|\vec{v}\| = 5$ so that when $\vec{v}$ is plotted in standard position, it lies in Quadrant II and makes a 60° angle with the negative x-axis.
2. For $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, find $\|\vec{v}\|$ and θ, $0 \leq \theta < 2\pi$ so that $\vec{v} = \|\vec{v}\| \langle \cos(\theta), \sin(\theta) \rangle$.
3. For the vectors $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$, find the following.

   (a) $\hat{v}$ (b) $\|\vec{v}\| - 2\|\vec{w}\|$ (c) $\|\vec{v} - 2\vec{w}\|$ (d) $\|\hat{w}\|$

SOLUTION:

1. We are told that $\|\vec{v}\| = 5$ and are given information about its direction, so we can use the formula $\vec{v} = \|\vec{v}\|\hat{v}$ to get the component form of $\vec{v}$. Sketching $\vec{v}$ we get:



So, $\hat{v} = \langle \cos(120°), \sin(120°) \rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$, so $\vec{v} = \|\vec{v}\|\hat{v} = 5\left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \left\langle -\frac{5}{2}, \frac{5\sqrt{3}}{2} \right\rangle$.

2. For $\vec{v} = \langle 3, -3\sqrt{3} \rangle$, we get $\|\vec{v}\| = \sqrt{(3)^2 + (-3\sqrt{3})^2} = 6$. To find θ, we need a Quadrant IV angle whose terminal side contains the point $(3, -3\sqrt{3})$. Going through the usual calculations, we find $\cos(\theta) = \frac{1}{2}$ and $\sin(\theta) = -\frac{\sqrt{3}}{2}$. Hence, $\theta = \frac{5\pi}{3}$. To check, we note that: $6\left\langle \cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right) \right\rangle = \langle 3, -3\sqrt{3} \rangle = \vec{v}$.

3. (a) Since we are given the component form of $\vec{v}$, we'll use the formula $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right)\vec{v}$. For $\vec{v} = \langle 3, 4 \rangle$, we have $\|\vec{v}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$. Hence, $\hat{v} = \frac{1}{5}\langle 3, 4 \rangle = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$.

   (b) We know from our work above that $\|\vec{v}\| = 5$, so to find $\|\vec{v}\| - 2\|\vec{w}\|$, we need only find $\|\vec{w}\|$. Since $\vec{w} = \langle 1, -2 \rangle$, we get $\|\vec{w}\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$. Hence, $\|\vec{v}\| - 2\|\vec{w}\| = 5 - 2\sqrt{5}$.

   (c) In the expression $\|\vec{v} - 2\vec{w}\|$, notice that the arithmetic on the vectors comes first, then the magnitude. We find $\vec{v} - 2\vec{w} = \langle 3, 4 \rangle - 2\langle 1, -2 \rangle = \langle 1, 8 \rangle$ so that $\|\vec{v} - 2\vec{w}\| = \|\langle 1, 8 \rangle\| = \sqrt{1^2 + 8^2} = \sqrt{65}$.

   (d) One approach to find $\|\hat{w}\|$, is to first find $\hat{w}$ and then take the magnitude.

   Using the formula $\hat{w} = \left(\frac{1}{\|\vec{w}\|}\right)\vec{w}$ along with $\|\vec{w}\| = \sqrt{5}$, which we found the in the previous problem, we get $\hat{w} = \frac{1}{\sqrt{5}}\langle 1, -2 \rangle = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle = \left\langle \frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5} \right\rangle$.

   Hence, $\|\hat{w}\| = \sqrt{\left(\frac{\sqrt{5}}{5}\right)^2 + \left(-\frac{2\sqrt{5}}{5}\right)^2} = \sqrt{\frac{5}{25} + \frac{20}{25}} = \sqrt{1} = 1$.

   Alternatively, since $\hat{w} = \left(\frac{1}{\|\vec{w}\|}\right)\vec{w}$, where $\frac{1}{\|\vec{w}\|} > 0$ is a scalar,

$$\|\hat{w}\| = \left\|\left(\frac{1}{\|\vec{w}\|}\right)\vec{w}\right\| = \frac{1}{\|\vec{w}\|}\|\vec{w}\| = \frac{\|\vec{w}\|}{\|\vec{w}\|} = 1.$$

   For a third way to show $\|\hat{w}\| = 1$, we note since $\hat{w} = \langle \cos(\theta), \sin(\theta) \rangle$ for some angle θ, $\|\hat{w}\| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1$, where we have used the Pythagorean Identity, $\cos^2(\theta) + \sin^2(\theta) = 1$.

EXAMPLE 5: A plane leaves an airport with an airspeed of 175 miles per hour with bearing N40°E. A 35 mile per hour wind is blowing at a bearing of S60°E. Find the true speed of the plane, rounded to the nearest mile per hour, and the true bearing of the plane, rounded to the nearest degree.

SOLUTION: Let $\vec{v}$ and $\vec{w}$ denote the velocity of the plane and wind, respectively. We set about finding $\vec{v} + \vec{w}$.

If we regard the airport as being at the origin, the positive y-axis acting as due north and the positive x-axis acting as due east, we see that the vectors $\vec{v}$ and $\vec{w}$ are in standard position and their directions correspond to the angles 50° and −30°, respectively.

Hence, the component form of $\vec{v} = 175\langle \cos(50°), \sin(50°) \rangle = \langle 175\cos(50°), 175\sin(50°) \rangle$ and the component form of $\vec{w} = \langle 35\cos(-30°), 35\sin(-30°) \rangle$.

Since we have no convenient way to express the exact values of cosine and sine of 50°, we leave both vectors in terms of cosines and sines and approximate the answer. Adding corresponding components, we find the resultant vector $\vec{v} + \vec{w} = \langle 175\cos(50^\circ) + 35\cos(-30^\circ), 175\sin(50^\circ) + 35\sin(-30^\circ)\rangle \approx \langle 142.8, 116.6\rangle$. To find the 'true' speed of the plane, we compute the magnitude of this resultant vector

$$\|\vec{v} + \vec{w}\| = \sqrt{(142.8)^2 + (116.6)^2} \approx 184$$

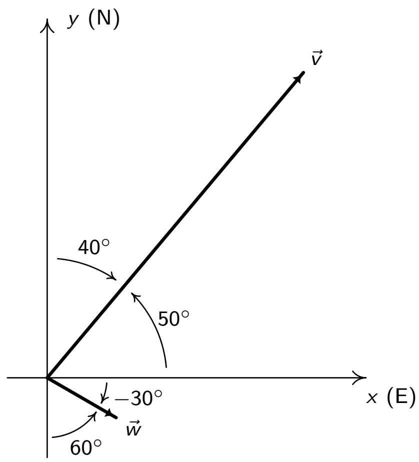
Hence, the 'true' speed of the plane is approximately 184 miles per hour.

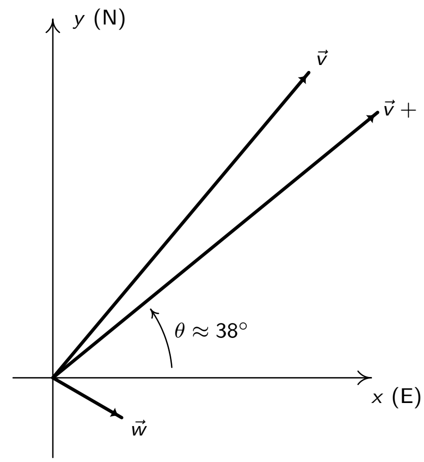
To find the true bearing, we need to find the angle θ whose terminal side when graphed in standard position contains $(x, y) \approx (142.8, 116.6)$.

Since both of these coordinates are positive, we know θ is a Quadrant I angle, as depicted below. Furthermore,

$$\tan(\theta) = \frac{y}{x} = \frac{116.6}{142.8},$$

so using the arctangent function, we get $\theta \approx 38^\circ$. Since, for the purposes of bearing, we need the angle between $\vec{v} + \vec{w}$ and the positive y-axis, we take the complement of θ and find the 'true' bearing of the plane to be approximately N52°E.



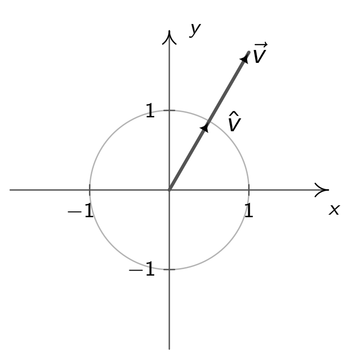


DEFINITION: Unit Vectors: Let $\vec{v}$ be a vector. If $\|\vec{v}\| = 1$, we say that $\vec{v}$ is a **unit vector**.

Note that if $\vec{v}$ is a unit vector, then necessarily, $\vec{v} = \|\vec{v}\|\hat{v} = 1 \cdot \hat{v} = \hat{v}$. Conversely, for any nonzero vector $\vec{v}$, $\|\hat{v}\| = 1$, so $\hat{v}$ is a unit vector. In other words, unit vectors are direction vectors and vice-versa. Indeed, the vector $\hat{v}$ which we have defined as 'the **direction** of $\vec{v}$' is also called 'the **unit vector in the direction** of $\vec{v}$.'

In practice, if $\vec{v}$ is a unit vector we write it as $\hat{v}$ as opposed to $\vec{v}$ because we have reserved the '$\hat{\ }$' notation for unit vectors. The process of multiplying a nonzero vector by the factor $\frac{1}{\|\vec{v}\|}$ to produce a unit vector is called '**normalizing** the vector.'

The terminal points of unit vectors, when plotted in standard position, lie on the Unit Circle. Hence, normalizing a nonzero vector $\vec{v}$ is tantamount to shrinking its terminal point, back to the Unit Circle.



Visualizing vector normalization $\hat{v} = \left(\frac{1}{\|\vec{v}\|}\right)\vec{v}$

Of all of the unit vectors, two deserve special mention.

DEFINITION: PRINCIPAL UNIT VECTORS

- The vector $\hat{\imath}$ is defined by $\hat{\imath} = \langle 1, 0 \rangle$
- The vector $\hat{\jmath}$ is defined by $\hat{\jmath} = \langle 0, 1 \rangle$

Geometrically, in the *xy*-plane, the vector $\hat{\imath}$ as represents the positive *x*-direction, whereas the vector $\hat{\jmath}$ represents the positive *y*-direction. We have the following 'decomposition' theorem.

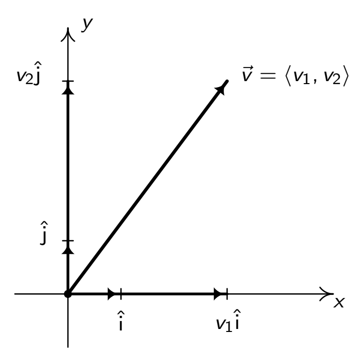
THEOREM: PRINCIPAL UNIT VECTOR DECOMPOSITION THEOREM:

Let $\vec{v}$ be a vector with component form $\vec{v} = \langle v_1, v_2 \rangle$. Then $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$.

The proof of this theorem is straightforward. Since $\hat{\imath} = \langle 1, 0 \rangle$ and $\hat{\jmath} = \langle 0, 1 \rangle$, we have from the definition of scalar multiplication and vector addition that

$$v_1\hat{\imath} + v_2\hat{\jmath} = v_1 \langle 1, 0 \rangle + v_2 \langle 0, 1 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle = \langle v_1, v_2 \rangle = \vec{v}$$

Geometrically, the situation looks like this:

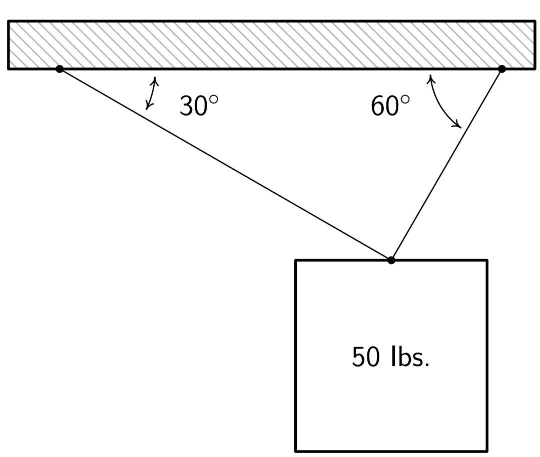


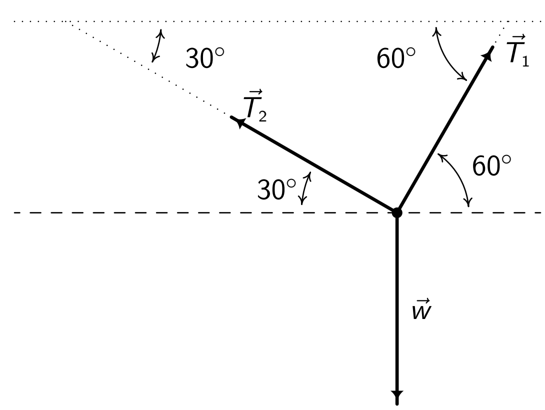
$$\vec{v} = \langle v_1, v_2 \rangle = v_1\hat{\imath} + v_2\hat{\jmath}.$$

We conclude this section with a classic example which demonstrates how vectors are used in physics to study forces. A 'force' is defined as a 'push' or a 'pull.' The intensity of the push or pull is the magnitude of the force, and is measured in Netwons (N) in the SI system or pounds (lbs.) in the English system.

EXAMPLE 6: A 50 pound speaker is suspended from the ceiling by two support braces. If one brace makes a 60° angle with the ceiling and the other makes a 30° angle with the ceiling, what are the tensions on each brace?

SOLUTION: We represent the problem schematically below along with the corresponding vector diagram.





We have three forces acting on the speaker: the weight of the speaker, which we'll call $\vec{w}$, pulling the speaker directly downward, and the forces on the support rods, which we'll call $\vec{T}_1$ and $\vec{T}_2$ (for 'tensions') acting upward at angles 60° and 30°, respectively.

We are looking for the tensions on the support, which are the magnitudes $\|\vec{T}_1\|$ and $\|\vec{T}_2\|$. In order for the speaker to remain stationary, we require $\vec{w}+\vec{T}_1+\vec{T}_2=\vec{0}$.

Viewing the common initial point of these vectors as the origin and the dashed line as the x-axis, we get component representations for the three vectors involved. We can model the weight of the speaker as a vector pointing directly downwards with a magnitude of 50 pounds. That is, $\|\vec{w}\|=50$ and $\hat{w}=-\hat{\jmath}=\langle 0,-1\rangle$. Hence, $\vec{w}=50\langle 0,-1\rangle=\langle 0,-50\rangle$. For the force in the first support, we get

$$\begin{aligned}\vec{T}_1 &= \|\vec{T}_1\|\left\langle \cos(60^\circ), \sin(60^\circ)\right\rangle \\ &= \left\langle \frac{\|\vec{T}_1\|}{2}, \frac{\|\vec{T}_1\|\sqrt{3}}{2}\right\rangle\end{aligned}$$

For the second support, we note that the angle 30° is measured from the negative x-axis, so the angle needed to write $\vec{T}_2$ in component form is 150°. Hence

$$\begin{aligned}\vec{T}_2 &= \|\vec{T}_2\|\left\langle \cos(150^\circ), \sin(150^\circ)\right\rangle \\ &= \left\langle -\frac{\|\vec{T}_2\|\sqrt{3}}{2}, \frac{\|\vec{T}_2\|}{2}\right\rangle\end{aligned}$$

The requirement $\vec{w}+\vec{T}_1+\vec{T}_2=\vec{0}$ gives us the vector equation:

$$\begin{aligned}\vec{w}+\vec{T}_1+\vec{T}_2 &= \vec{0} \\ \langle 0,-50\rangle + \left\langle \frac{\|\vec{T}_1\|}{2}, \frac{\|\vec{T}_1\|\sqrt{3}}{2}\right\rangle + \left\langle -\frac{\|\vec{T}_2\|\sqrt{3}}{2}, \frac{\|\vec{T}_2\|}{2}\right\rangle &= \langle 0,0\rangle \\ \left\langle \frac{\|\vec{T}_1\|}{2} - \frac{\|\vec{T}_2\|\sqrt{3}}{2}, \frac{\|\vec{T}_1\|\sqrt{3}}{2} + \frac{\|\vec{T}_2\|}{2} - 50\right\rangle &= \langle 0,0\rangle\end{aligned}$$

Equating the corresponding components of the vectors on each side, we get a system of linear equations in the variables $\|\vec{T}_1\|$ and $\|\vec{T}_2\|$.

$$\begin{cases} (E1) \quad \dfrac{\|\vec{T}_1\|}{2} - \dfrac{\|\vec{T}_2\|\sqrt{3}}{2} = 0 \\ (E2) \quad \dfrac{\|\vec{T}_1\|\sqrt{3}}{2} + \dfrac{\|\vec{T}_2\|}{2} - 50 = 0 \end{cases}$$

From $(E1)$, we get $\|\vec{T}_1\|=\|\vec{T}_2\|\sqrt{3}$. Substituting that into $(E2)$ gives $\frac{(\|\vec{T}_2\|\sqrt{3})\sqrt{3}}{2}+\frac{\|\vec{T}_2\|}{2}-50=0$.

Solving, we get $2\|\vec{T}_2\|-50=0$, so $\|\vec{T}_2\|=25$ pounds. Hence, $\|\vec{T}_1\|=\|\vec{T}_2\|\sqrt{3}=25\sqrt{3}$ pounds.

Note that the sum of the tensions on the wires in this last example exceed the 50 pounds of the speaker. Why?

HOMEWORK: Section 13.1: 13 - 73 every other odd; 75*, 77 - 81*, 85*, 88*